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1976 J. Phys. A: Math. Gen. 9 1637

(http://iopscience.iop.org/0305-4470/9/10/015)

J. Phys. A: Math. Gen., Vol. 9, No. 10, 1976. Printed in Great Britain. © 1976.

The field equations of a conformally invariant, scalar–tensor theory

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Received 22 March 1976, in final form 24 May 1976

Abstract. A general, conformally invariant theory of the general relativistic, gravitational field coupled with a scalar field is considered. The field equations of the theory are derived from a variational action principle and also from the assumption of the geodesic motion of test particles. These are shown to be identical. The scalar field is assumed to be a long-range field with the view to establishing a cosmological theory. An upper bound for the background scalar field is estimated.

1. Introduction

The only empirically observed, long-range fields are those corresponding to spin $S=\frac{1}{2}$, 1 and 2, that is, the neutrino, the electromagnetic and the gravitational fields respectively. It is difficult to obtain consistent field equations for a long-range field with $S=\frac{3}{2}$, but not for a scalar field with $S=0$. In fact, such a (scalar) field, determined by an homogeneous wave equation, has been suggested in connection with the Brans–Dicke cosmology (Brans and Dicke 1961), though it has not been hitherto observed.

There are several reasons why one should require the scalar field ϕ coupling with a general relativistic gravitational field g_{ij} (Latin indices i, j going from 1 to 4) to be conformally invariant, that is, invariant under the transformations

$$g'_{ij}=\mathrm{e}^{-\sigma}g_{ij}, \qquad \phi'=\mathrm{e}^{-\beta\sigma}\phi, \tag{1}$$

where σ is a scalar function of the space–time coordinates x^i, and β is a constant. The gravitational field equations can be written in a conformally invariant way and the neutrino and electromagnetic fields are known to be conformal. One would expect the quanta of a zero rest mass field to move along null geodesics

$$\mathrm{d}s=0, \tag{2}$$

which remain invariant under all conformal mappings.

It has been shown by Penrose (1965) that a conformally invariant wave equation is, with $\beta=\frac{1}{2}$,

$$\Box\phi+\tfrac{1}{6}R\phi=0, \tag{3}$$

where $\Box\phi\equiv g^{ij}\phi_{;ij}$; $R=g^{ij}R_{ij}$, and R_{ij} is the Ricci tensor while semicolons denote, as usual, covariant differentiation with respect to the coordinates.

In fact, under a transformation (1), we have

$$\Box\phi + AR\phi + B\phi^n = e^{(\beta-1)\sigma}[\Box'\phi' + \beta(\sigma'^k{}_{;k} + (1+\beta)\sigma_{,k}\sigma'^k)\phi' + (2\beta+1)\sigma_{,k}\sigma'^k + AR'\phi' + 3A\sigma'^k{}_{;k}\phi' + \tfrac{3}{2}\sigma_{,k}\sigma'^k\phi'] + B\,e^{n\beta\sigma}\phi'^n, \tag{4}$$

A, B, n constant. Hence

$$\Box\phi + AR\phi + B\phi^n = e^{(\beta-1)\sigma}(\Box'\phi' + AR'\phi' + B\phi'^n)$$

if and only if, $A = \frac{1}{6}$, $\beta = -\frac{1}{2}$, $n = 3$, so that, the most general inhomogeneous, conformally invariant, wave equation is given by

$$\Box\phi + \tfrac{1}{6}R\phi + \alpha\phi^3 = 0, \tag{5}$$

where α is a constant. If need be, one can further insert on the right-hand side of equation (5) a source term, say T/λ.

Variational derivation of such wave equations coupled to gravitational fields has been discussed by Freund (1974), and Bramson (1974), whose matter Lagrangian was constructed from a Dirac field ψ.

In this paper we are going to study a general, conformal scalar field coupling consistent with general relativistic theory. Our aim is to relate the resulting field equations to cosmological considerations. Some of these will be discussed in a subsequent publication.

2. Phenomenological derivation of the field equations

Consider the field equations

$$fG_{ij} + 8\pi T_{ij} + A\phi_{,i}\phi_{,j} + Bg_{ij}\phi_{,k}\phi^{,k} + C\phi_{;ij} + Dg_{ij}\Box\phi + Eg_{ij} = 0, \tag{6}$$

$$\Box\phi + \tfrac{1}{6}R\phi + \alpha\phi^3 = 8\pi T/\lambda, \tag{7}$$

where f, A, B, C, D and E are functions of ϕ only, $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$, is the Einstein tensor, T_{ij} is the energy–stress–momentum tensor of matter, $T = g^{ij}T_{ij}$, and α, λ are constants.

It is well known that

$$\phi_{,j}G_i^j = (\Box\phi)_{,i} - \Box\phi_{,i} - \tfrac{1}{2}R\phi_{,i}. \tag{8}$$

Also, contracting equations (6) with the help of g^{ij}:

$$-fR + 8\pi T + (A+4B)\phi_{,k}\phi^{,k} + (C+4D)\Box\phi + 4E = 0, \tag{9}$$

or eliminating T between equations (7) and (9), and solving for R

$$R = \frac{(C+4D+\lambda)\Box\phi + (A+4B)\phi_{,k}\phi^{,k} + \alpha\lambda\phi^3 + 4E}{f - \frac{1}{6}\lambda\phi}, \tag{10}$$

providing

$$f \neq \tfrac{1}{6}\lambda\phi. \tag{11}$$

Raising the index j in (6), taking the divergence, using equations (8) and (9) and requiring that T_i^j should be conserved,

$$T_{i\ ;j}^{j} \equiv 0, \tag{12}$$

we easily find that

$$f'[(\Box\phi)_{,i} - \Box\phi_{,i} - \tfrac{1}{2}R\phi_{,i}] + A'\phi_{,i}\phi_{,k}\phi^{,k} + B'\phi_{,i}\phi_{,k}\phi^{,k} + C'\phi_{,k}\phi_{,i}^{\ ;k}$$
$$+ D'\phi_{,i}\Box\phi + E'\phi_{,i} + A\phi_{;ik}\phi^{,k} + A\phi_{,i}\phi^{,k}_{\ ;k}$$
$$+ B\phi_{;ki}\phi^{,k} + B\phi_{,k}\phi^{,k}_{\ ;i} + C\Box\phi_{,i} + D(\Box\phi)_{,i} = 0, \tag{13}$$

primes denoting differentiation with respect to ϕ. Equation (13) must be satisfied identically. Eliminating from it R with the help of (10), and equating to zero the coefficients of $(\Box\phi)_{,i}$, $\Box\phi_{,i}$, $\phi_{,i}\Box\phi$, $\phi_{,i}\phi_{,k}\phi^{,k}$ and $\phi_{;ik}\phi^{,k}$, respectively, we obtain the following six ordinary differential equations:

$$f' + D = 0, \tag{14a}$$

$$f' - C = 0, \tag{14b}$$

$$\frac{3f'}{\lambda\phi - 6f}(C + 4D + \lambda) + A + D' = 0, \tag{14c}$$

$$\frac{3f'}{\lambda\phi - 6f}(A + 4B) + A' + B' = 0, \tag{14d}$$

$$C' + A + 2B = 0, \tag{14e}$$

$$\frac{3f'}{\lambda\phi - 6f}(\alpha\lambda\phi^3 + 4E) + E' = 0. \tag{14f}$$

Equations (14) serve to determine the six unknown functions f, A, B, C, D and E of ϕ consistently with the equations (6) and (7). We may notice that the first five of these functions are determined by equations (14a) to (14e), and that once these are solved (14f) will determine E.

3. Derivation of the field equations continued

Let us solve first the simultaneous equations (14a) to (14e). Let

$$h = \lambda\phi - 6f \neq 0. \tag{15}$$

Then from (14a, b, c and e),

$$C = -F = \tfrac{1}{6}(\lambda - h'),$$
$$A = -\tfrac{1}{6}h'' - \tfrac{1}{4}(\lambda^2 - h'^2)/h,$$
$$B = \tfrac{1}{6}h'' + \tfrac{1}{8}(\lambda^2 - h'^2)/h,$$

and, substituting into (14d) and simplifying,

$$2hh'' + \lambda^2 - h'^2 = 0. \tag{16}$$

Let

$$u = 1/h' = \mathrm{d}\phi/\mathrm{d}h, \qquad \dot{u} = \mathrm{d}u/\mathrm{d}h. \tag{17}$$

Equation (17) then reduces to

$$2h\dot{u} = u(\lambda^2 u^2 - 1). \tag{18}$$

Integrating equation (18) twice, we obtain

$$h = \lambda^2 a - \frac{(\phi + \phi_0)^2}{4a}, \tag{19}$$

where a and ϕ_0 are constants of integration, and

$$f = \tfrac{1}{6}\left(-\lambda^2 a + \lambda\phi + \frac{(\phi+\phi_0)^2}{4a}\right) = -\frac{1}{24a}[8\lambda^2 a + 4a\lambda\phi_0 - (\phi + \phi_0 + 2\lambda a)^2]. \tag{20}$$

Also

$$\frac{3f'}{h} = \frac{3f'}{\lambda\phi - f} = \frac{1}{2a}\,\frac{\lambda + [(\phi+\phi_0)/2a]}{\lambda^2 - [(\phi+\phi_0)^2/4a^2]} = \frac{1}{2\lambda a - \phi_0 - \phi}.$$

Equation (14f) can therefore be written in the form

$$\frac{\mathrm{d}E}{\mathrm{d}\psi} - \frac{4E}{\psi} - \frac{\alpha\lambda(\gamma-\psi)^3}{\psi} = 0, \tag{21}$$

where $\psi = 2a\lambda - \phi_0 - \phi$, $\gamma = 2a\lambda - \phi_0$.

Hence, if K is a third constant of integration,

$$E = \alpha\lambda\left(\frac{K}{\alpha\lambda}\psi^4 + \psi^3 - \frac{3\gamma}{2}\psi^2 + \gamma^2\psi - \frac{\gamma^3}{4}\right). \tag{22}$$

This result is compatible with a conformally invariant wave equation if E is a power (fourth) of a linear function of ϕ, and hence, only if

$$K = -\alpha\lambda/4\gamma,$$

when

$$E = -(\alpha\lambda/4\gamma)\phi^4. \tag{23}$$

It follows that, apart from the constants of integration a and ϕ_0 the functions f, A, B, C, D and E are uniquely determined from the requirement that T_{ij} should be conserved. The gravitational field equations coupled to the scalar field are

$$-\frac{1}{24a}[4a\lambda\phi' - (\phi+\phi')^2]G_{ij} + 8\pi T_{ij} + \frac{1}{24a}(g_{ij}\phi_{,k}\phi^{,k} - 4\phi_{,i}\phi_{,j})$$
$$+\frac{1}{12a}(\phi+\phi')(\phi_{;ij} - g_{ij}\Box\phi) - \frac{\alpha\lambda}{4\gamma}\phi^4 g_{ij} = 0, \tag{24}$$

where $\phi' = \phi_0 + 2a\lambda$.

4. Variational derivation of the field equations

We show now that equations (24) and (7) can be derived from a single variational action principle. Consider first an action principle of the form

$$\delta\int (Fg^{ij}R_{ij} + Mg^{ij}\phi_{,i}\phi_{,j} + N - 8\pi L_m)\sqrt{-g}\,\mathrm{d}^4x = 0, \tag{25}$$

where F, M and N are scalar functions of ϕ only and L_m denotes a matter term, or source action of the field to be varied with respect to g_{ij} only. The energy–stress–momentum tensor T_{ij} is proportional to the Hamiltonian derivative of L_m with respect to g^{ij}, say

$$T_{ij} = hL_m / hg^{ij}.$$

Hence, with the usual conditions on the boundary of the region of integration, we obtain

$$\int \{[(FR_{ij} + M\phi_{,i}\phi_{,j}) - \tfrac{1}{2}g_{ij}(FR + M\phi_{,k}\phi^{,k} + N) + (F_{;ij} - g_{ij}g^{kl}F_{;kl}) - 8\pi T_{ij}]\delta g^{ij}$$
$$+ [F'R + M'\phi_{,k}\phi^{,k} - 2M\Box\phi + N']\delta\phi\}\sqrt{-g}\, d^4x = 0. \tag{26}$$

Since

$$F_{;ij} = F''\phi_{,i}\phi_{,j} + F'\phi_{;ij}, \qquad g^{kl}F_{;kl} = F''\phi_{,k}\phi^{,k} + F'\Box\phi,$$

and the variations in g^{ij} and ϕ are assumed to be independent, we get the field equations

$$FG_{ij} + F'(\phi_{;ij} - g_{ij}\Box\phi) + (F'' + M)\phi_{,i}\phi_{,j} - (F'' + \tfrac{1}{2}M)g_{ij}\phi_{,k}\phi^{,k} - \tfrac{1}{2}Ng_{ij} - 8\pi T_{ij} = 0, \tag{27}$$

and

$$F'R + M'\phi_{,k}\phi^{,k} - 2M\Box\phi + N' = 0. \tag{28}$$

Comparing equations (27) and (24), we get

$$F = \frac{1}{24a}[4a\lambda\phi' - (\phi + \phi')^2], \qquad M = \frac{1}{4a}, \qquad N = \frac{\alpha\lambda}{2\gamma}\phi^4. \tag{29}$$

We must next eliminate the term $\phi' R$ from (28) which now reads

$$\frac{\phi + \phi'}{12a}R + \frac{1}{2a}\Box\phi + \frac{2\alpha\lambda}{\gamma}\phi^3 = 0. \tag{30}$$

Also, contracting equation (27) (or (24)) with g^{ij},

$$\frac{1}{24a}[4a\lambda\phi' - (\phi + \phi')^2]R - \frac{1}{4a}(\phi + \phi')\Box\phi - \frac{\alpha\lambda}{\gamma}\phi^4 + 8\pi T = 0. \tag{31}$$

Hence, multiplying equation (30) by $\frac{1}{2}(\phi + \phi')$and adding to equation (31):

$$\tfrac{1}{6}\phi' R = -\frac{8\pi T}{\lambda} - \frac{\alpha\phi'}{\gamma}\phi^3 \tag{32}$$

so that, substituting R into equation (31) for $\phi' R$, from equation (32), and putting $4a\lambda = \gamma + \phi'$,

$$\Box\phi + \tfrac{1}{6}\phi R + \alpha\phi^3 = 8\pi T/\lambda,$$

that is, equations (30) and (7) are equivalent. This completes our variational derivation of the field equations with the consequence that they are, in general, compatible. We shall obtain some cosmological solutions to these equations in the subsequent paper already mentioned.

It is convenient to eliminate a factor $2\lambda/\gamma$ from the Lagrangian so that, for the free field, it becomes

$$\left[\frac{\gamma\phi'}{12}-\frac{\gamma}{12(\phi'+\gamma)}(\phi+\phi')^2\right]R+\frac{\gamma}{2(\phi'+\gamma)}\phi_{,k}\phi^{,k}-\frac{\alpha}{4}\phi^4. \tag{33}$$

The factor $\gamma/2\lambda$ may be absorbed into the matter term L_m.

If we now introduce constants μ, ν and q by

$$\gamma\phi'/12=\mu^2, \qquad \gamma/(\gamma+\phi')=q, \qquad 12\mu\nu/q=\phi', \tag{34}$$

so that $q=1-12\nu^2$, the free-field Lagrangian becomes

$$\left[\frac{\mu^2}{1-12\nu^2}-\frac{1-12\nu^2}{12}\left(\phi+\frac{12\mu\nu}{1-12\nu^2}\right)^2\right]R+\frac{1}{2}(1-12\nu^2)\phi_{,k}\phi^{,k}-\frac{\alpha}{4}\phi^4, \tag{35}$$

or

$$(\mu^2-2\mu\nu\phi-\tfrac{1}{12}q\phi^2)R+\tfrac{1}{2}q\phi_{,k}\phi^{,k}-\tfrac{1}{4}\alpha\phi^4. \tag{36}$$

In the special case therefore, when

$$q=0, \tag{37}$$

the action acquires the form

$$\int(\psi R-\tfrac{1}{4}\alpha\phi^4-8\pi L_m)\sqrt{-g}\,d^4x, \tag{38}$$

where $\psi=\mu^2-2\mu\nu\phi$. The quartic term $-\frac{1}{4}\alpha\phi^4$, has the form of a (variable) cosmological term and plays no part in the empirical test of gravitational theory based on observations within the solar system. However, if this term is omitted, variation of (38) leads to the special case of the Brans–Dicke theory with vanishing coupling constant. This particular case of the theory is known to be contradicted by observational evidence. Hence, and also to obtain the correct sign of the $\phi_{,k}\phi^{,k}$ term, we must require that

$$q=1-12\nu^2>0. \tag{39}$$

This allows us to change the constants and the scalar field by $\tilde{\phi}=\gamma\phi$, $\tilde{\mu}=\gamma\mu$, $\gamma^2=|\alpha|$, and to write the field equations in the final form (γ is not the same γ as previously used):

$$[\tilde{\mu}^2-\tfrac{1}{12}(\tilde{\phi}+12\tilde{\mu}\nu)^2]G_{ij}+\tfrac{1}{3}\tilde{\phi}_{,i}\tilde{\phi}_{,j}-\tfrac{1}{12}g_{ij}\tilde{\phi}_{,k}\tilde{\phi}^{,k}+\tfrac{1}{6}(\tilde{\phi}+12\tilde{\mu}\nu)(g_{ij}\Box\tilde{\phi}-\tilde{\phi}_{;ij})$$
$$+(\epsilon\tilde{\phi}^4/8q)g_{ij}=-8\pi\gamma^2T_{ij} \tag{40}$$

$$\Box\tilde{\phi}+\tfrac{1}{6}R\tilde{\phi}+\epsilon\tilde{\phi}^3=-(16\pi\nu\gamma^2/\tilde{\mu})T \tag{41}$$

$$\tilde{\mu}^2R=8\pi\gamma^2T-(6\epsilon\tilde{\mu}\nu/q)\tilde{\phi}^3, \tag{42}$$

where $\epsilon=\pm1$ or 0, depending on the sign of α (α, of course, can be zero).

5. Consequences of the scalar field coupling

If the scalar field is constant, equations (40) are equivalent to Einstein's gravitational field equations with a cosmological constant. On the other hand, equations (41) and (42) admit a solution

$$\tilde{\phi}=\text{constant} \tag{43}$$

for an arbitrary distribution of matter only if α vanishes. If

$$\alpha \neq 0, \tag{44}$$

such a solution is possible providing T is constant. This is an unreasonable restriction. Hence, if we are to achieve a genuine extension of Einstein's theory, inclusion of the non-zero parameter α is necessary.

Incidentally, when $\tilde{\phi}$ vanishes, the wave equation implies that the energy–stress–momentum tensor is traceless so that we have an electromagnetic field of the Einstein–Maxwell theory. Furthermore, equation (40) implies the dependence of the 'gravitational constant' G on the $\tilde{\phi}$ field, of the form

$$\frac{G}{c^4} \simeq \frac{\gamma^2}{\mu^2 - \frac{1}{12}(\phi + 12\mu\nu)^2}. \tag{45}$$

Hence, for a variable $\tilde{\phi}$ field, G likewise is variable. This is a necessary consequence of the scalar field coupling. Similarly, comparison between equation (43) and the general relativistic expression

$$R = \frac{8\pi G}{c^2} T - 4\Lambda, \tag{46}$$

shows that the cosmological 'constant' Λ is given by

$$\Lambda = \frac{3\epsilon\nu}{2\tilde{\mu}q} \tilde{\phi}^3. \tag{47}$$

Since the equations (40) and (41) are invariant if the sign of $\tilde{\phi}$ and of ν is simultaneously changed, the sign of the pure number ν is arbitrary, and we shall assume in the second paper already mentioned that

$$\nu < 0. \tag{48}$$

Let us consider next, in a background cosmological space–time, a perturbation to the background scalar field. Let us put

$$\tilde{\phi} = \tilde{\phi}_0 + \tilde{\phi}_1 \tag{49}$$

with $\tilde{\phi}_1 \ll \tilde{\phi}_0$. The approximate wave equation for $\tilde{\phi}_1$ is

$$\Box\tilde{\phi}_1 + (\tfrac{1}{6}R_0 + 3\epsilon\tilde{\phi}_0^2)\tilde{\phi}_1 = 0 \tag{50}$$

where R_0 is the background value of the scalar curvature. The above equation will possess causality-preserving solutions if the sign of $\frac{1}{6}R_0 + 3\epsilon\tilde{\phi}_0^2$ is negative. If R_0 is negative as is the usual case in cosmological models in general relativity this condition is guaranteed if $\epsilon = -1$. It is possible that R_0 may be positive in this theory (cf equation (42)) in which case both the conditions $\epsilon = -1$ and $\tilde{\phi}_0^2 > \frac{1}{2}R_0$ are required. Hence in the sequel we shall assume that $\epsilon = -1$ although the above argument shows that causality-preserving solutions of (50) are possible if $\epsilon = +1$.

Since equation (50) is a Klein–Gordon equation it predicts an effective range for the scalar field of the order of $(3\phi_0^2 - \frac{1}{6}R_0)^{-1/2}$. Thus a large value of the background scalar field is incompatible with its assumed long-range nature and ϕ_0 should be small in comparison to the radius of the universe.

6. Conformal invariance of the gravitational and scalar fields

Conformal invariance was used to justify the adoption of the wave equation (5) for the scalar field, yet it is evident that the gravitational field equations (41) are not themselves conformally invariant. However, it may be shown that the field equations presented here are those of a conformally invariant theory expressed in a particular gauge. We are also able to shed further light on the assumption of the independence of the source Lagrangian upon ϕ.

In connection with the former point it has been shown that general relativity itself is a conformally invariant theory expressed in a particular gauge (cf Bicknell 1976). The conformally invariant action for general relativity (without a cosmological term) may be expressed in the form

$$I=\int(\tfrac{1}{12}\omega^2R-\tfrac{1}{2}\omega_{,i}\omega^{,i}-8\pi L_{\mathrm{m}})\sqrt{-g}\,\mathrm{d}^4x \tag{51}$$

where ω is a conformally invariant scalar field. The above action is conformally invariant and reduces to that of general relativity in the gauge in which ω is a constant. Now consider the action

$$I=\int[\tfrac{1}{12}(\omega^2-\phi^2)R+\tfrac{1}{2}(\phi_{,i}\phi^{,i}-\omega_{,i}\omega^{,i})-\tfrac{1}{4}\alpha\phi^4-8\pi L_{\mathrm{m}}]\sqrt{-g}\,\mathrm{d}^4x. \tag{52}$$

The free-field part of this action is also conformally invariant under the transformations

$$g_{ij}=\mathrm{e}^{\sigma}g'_{ij} \tag{53}$$

$$\phi=\mathrm{e}^{-\sigma/2}\phi' \qquad \omega=\mathrm{e}^{-\sigma/2}\omega'. \tag{54}$$

The requirement of the invariance of

$$I_{\mathrm{m}}=\int L_{\mathrm{m}}\sqrt{-g}\,\mathrm{d}^4x$$

under the infinitesimal coordinate transformation

$$x^i=x'^i+\delta\xi^i \tag{55}$$

and the infinitesimal conformal transformation

$$g_{ij}=(1+\delta\sigma)g'_{ij} \tag{56}$$

yields

$$T^{ij}{}_{;j}=\frac{1}{2}S_1\frac{\omega^{,i}}{\omega}+\frac{1}{2}S_2\frac{\phi^{,i}}{\phi} \tag{57}$$

$$T=\tfrac{1}{2}S_1\omega+\tfrac{1}{2}S_2\phi \tag{58}$$

where T^{ij}, S_1 and S_2 are defined by

$$\delta I_{\mathrm{m}}=\int(T^{ij}\delta g_{ij}+S_1\delta\omega+S_2\delta\phi)\sqrt{-g}\,\mathrm{d}^4x. \tag{59}$$

We now make the assumption that the source terms are proportional, say

$$S_2=2\sqrt{3}\nu S_1. \tag{60}$$

Since the scalar $\omega + 2\sqrt{3}\nu\phi$ transforms as ω or ϕ, a space may be chosen so that

$$\omega + 2\sqrt{3}\nu\phi = \text{constant} = 2\sqrt{3}\mu, \tag{61}$$

say. Then

$$\omega = 2\sqrt{3}\mu\left(1 - \frac{\nu\phi}{\mu}\right) \tag{62}$$

and substitution of (62) and (60) into equation (57) implies that

$$T^{ij}{}_{;j} = 0. \tag{63}$$

The equations of motion of a test particle are therefore geodesics of the space–time. In the particular space considered, that selected by the gauge condition (61), the free-field Lagrangian in the action integral (52) becomes that given by (36).

It remains to justify the assumption made in § 4 that one can neglect the dependence of the matter term upon ϕ. This in fact follows from our assumption (60) relating the source term of the ω and ϕ fields. This assumption together with the gauge condition (61) implies that

$$S_1\delta\omega + S_2\delta\phi = 0 \tag{64}$$

and therefore, referring to equation (59) there is no contribution to δI_m from $\delta\phi$.

Thus it has been shown that the equations (40) and (41) of the scalar–tensor theory considered in § 4 of this paper are in fact the equations of a conformally invariant theory expressed in a particular gauge. To make the theory manifestly conformally invariant it is necessary to introduce a further conformally invariant scalar field (ω) into the framework which may be eliminated by the choice of a particular gauge. The gauge chosen here is in fact a sensible one since in it stress energy is conserved and the equations of motion are geodesics.

It is of interest to consider here the equations for the scalar fields ϕ in a general gauge. They are

$$\Box\omega + \tfrac{1}{6}R\omega = -8\pi S_1$$

$$\Box\phi + \tfrac{1}{6}R\phi + \alpha\phi^3 = -8\pi S_2.$$

It can be seen here why the inclusion of the ϕ^3 term is necessary once the assumption has been made on the proportionality of the two source terms. If the ϕ^3 term were absent one could find solutions such that ω and ϕ were proportional. One could then choose a gauge in which both were constant and the theory would collapse to general relativity. Another way of introducing an asymmetry between the two scalar fields would be to introduce different source terms but that will not be considered in the present context.

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